North-Holland Publishing Company

COMPLETE SUBGRAPHS OF THE GRAPHS OF CONVEX POLYTOPES

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Received 31 July 1979

It is shown that if three vertices of the graph $G(P)$ of a convex 3-polytope P are chosen, then $G(P)$ contains a refinement of the complete graph C_4 on four vertices, for which the three chosen vertices are principal (that is, correspond to vertices of C_4 in the refinement). In general, all four vertices may not be preassigned as principal. For dimensions $d \geqslant 4$, simple (simplicial) d-polytopes are constructed whose graphs contain sets of three (four) vertices, which cannot all be principal in any refinement of C_{d+1}.

1. Introduction

The *graph* of a d-polytope (d-dimensional convex polytope) P is the complex $G(P)$ consisting of the vertices and edges of P. The *complete graph* C_n on n vertices is the graph with n vertices, every pair of which is joined by an edge. The graph of the d-simplex T^d is isomorphic to C_{d+1}. All the graphs we consider here may be regarded as embedded in some euclidean space.

A cell-complex B is a *refinement* of another complex A if there is a homeomorphism $f: B \to A$ such that, for each cell C of A, $f^{-1}(C)$ is a union of cells of B. The homeomorphism f is called a *refinement map* from B to A. A cell D of B is called *principal if* $f(D)$ is a cell of A. In the particular case of graphs, note that the vertices of a principal edge will themselves be principal. If G contains a subgraph which is a refinement of a graph H, we say that G contains a refinement of H.

Grünbaum ([1]; see also [2, p. 200]) has shown, as a corollary of a stronger result, that the graph $G(P)$ of a d-polytope P contains a refinement of C_{d+1}. Furthermore, if any edge E of P is chosen, a refinement of C_{d+1} may be found for which E, and hence also its vertices, are principal. Since C_{d+1} is isomorphic to $G(T^d)$, C_{d+1} is the largest complete graph which is a refiement in the graph of every d-polytope.

In this paper, we consider the problem of determining how many vertices of a d-polytope P can be preassigned as principal in such a refiement of C_{d+1}. Grünbaum's result implies that we can preassign as principal any vertex of P, or any two vertices if they happen to be endpoints of the same edge. Larman and

Mani [3] show that if P is simplicial, two vertices of P can be preassigned as principal, however they are situated (we shall discuss this in Section 3).

If P is a d-polytope, define $r(P)$ to be the largest positive integer k such that, if any k vertices of P are chosen, there is a refinement of C_{d+1} in $G(P)$ such that all the chosen vertices are principal. Further define

$$r(d) = \min\{r(P) \mid P \in \mathscr{P}^d\}$$

and

$$r_s(d) = \min\{r(P) \mid P \in \mathscr{P}^d_s\},$$

where $\mathscr{P}^d$ ($\mathscr{P}^d_s$) is the family of all (simplicial) d-polytopes. Of course, $r(d) \leqslant r_s(d)$.

A 1-polytope is a line segment; hence $r(1) = r_s(1) = 2$.

The graph of a 2-polytope (convex polygon) is a simple circuit with n edges, for some $n \geqslant 3$. The complete graph C_3 is a circuit of three edges. It is thus clear that $G(P)$ is a refinement of C_3, and three vertices can always be preassigned. Thus $r(2) = r_s(2) = 3$.

In Section 2 of this paper, we show that $r(3) = r_s(3) = 3$. In Section 3, we construct examples to show that for $d \geqslant 4$, $r(d) \leqslant 2$ and $r_s(d) \leqslant 3$. (Larman and Mani give in [3] an example of a simplical 3-polytope P with $r(P) < 4$, but their supposed examples in higher dimensions are not simplicial.) The table below summarizes our present knowledge of $r(d)$ and $r_s(d)$. The authors' guess is that for $d \geqslant 4$, $r(d) = 2$ and $r_s(d) = 3$.

d	$r(d)$	$r_s(d)$
1	2	2
2	3	3
3	3	3
$\geqslant 4$	1 or 2	2 or 3

2. Three dimensions

Easy examples show that $r(3) \leqslant 3$; for instance, the four vertices of a square face of a triangular prism cannot all be principal. The main result of this paper shows that there is, in fact, equality here.

Theorem 1. *Let x_1, x_2 and x_3 be distinct vertices of a 3-polytope P. Then $G(P)$ contains a refinement of C_4, for which x_1, x_2 and x_3 are three of the principal vertices.*

Proof. We prove Theorem 1 in two stages. We first show that there are three arcs in $G(P)$ from x_1 to x_2, and disjoint except at x_1 and x_2, one of which passes through x_3. Since $G(P)$ is planar, we can think of $G(P)$ as embedded in the euclidean plane E^2. Since $G(P)$ is 3-connected, there are arcs Γ_1, Γ_2 and Γ_3 in $G(P)$ joining x_1 and x_2, and disjoint except at their endpoints. If $x_3 \in$

$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma$, say, we are done; thus we may assume that x_3 lies in one of the connected components of $E^2 \backslash \Gamma$, say the component bounded by $\Gamma_2 \cup \Gamma_3$. Since $G(P)$ is 3-connected, there exist arcs Λ_1, Λ_2 and Λ_3 in $G(P)$ from x_3 to $\Gamma_2 \cup \Gamma_3$, disjoint except at x_3. We may suppose that for $i = 1, 2, 3$, $\Lambda_i \cap (\Gamma_2 \cup \Gamma_3)$ is a single point y_i. One of the arcs Γ_2 or Γ_3 must meet two of the arcs Λ_1, Λ_2 or Λ_3; without loss of generality, suppose that Γ_3 meets Λ_1 and Λ_2, and that y_1 is nearer to x_1 on Γ_3 than y_2. Writing $\Delta(u, v)$ for that part of the arc Δ between its two vertices u and v, we then see that we can replace Γ_3 by

$$\Gamma_3' = \Gamma_3(x_1, y_1) \cup \Lambda_1 \cup \Lambda_2 \cup \Gamma_3(y_2, x_2),$$

noting that Γ_3' meets Γ_1 or Γ_2 only in x_1 and x_2.

Notice that this conclusion does not hold for general 3-connected graphs, such as the complete bipartite graph on two groups of three vertices.

So, changing Γ_3' back to Γ_3, we have arcs Γ_i $(i = 1, 2, 3)$ from x_1 to x_2 in $G(P)$, disjoint except at their endpoints, with $x_3 \in \Gamma_3$. Using the 3-connectedness of $G(P)$ again, there are arcs Δ_i $(i = 1, 2, 3)$ in $G(P)$, disjoint except at x_3, leading from x_3 to $\Gamma_1 \cup \Gamma_2$. We may suppose the Δ_i so chosen, that the number of edges in $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$, which are not already in $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, is minimal; that is, we introduce as few new edges as possible. Then we claim that each of x_1 and x_2 lies in Δ. For, if (say) $x_1 \notin \Delta$, let w_1 be the nearest vertex of Γ_3 to x_1 on Δ; suppose that $w_1 \in \Delta_1$, and if $w_1 = x_3$, further suppose that $\Delta_1 \neq \Gamma_3(x_3, x_2)$. We can then replace Δ_1 by $\Delta_1(x_3, w_1) \cup \Gamma_3(w_1, x_1)$, which gives a contradiction to our minimality assumption. Thus $x_i \in \Delta$ $(i = 1, 2)$, and it is now clear that $\Gamma_1 \cup \Gamma_2 \cup \Delta$ is the required refinement, which completes the proof of Theorem 1.

We have thus shown that $r(3) = 3$. The example of Theorem 4 in Section 3 will show that $r_s(3) \leqslant 3$, and so $r_s(3) = 3$ also.

3. Four or more dimensions

In contrast to Theorem 1, in dimensions higher than three we have the following result.

Theorem 2. *Let $d \geqslant 4$. Then there is a simple d-polytope P, with some three vertices which cannot all be principal for any refinement of C_{d+1} in $G(P)$.*

Proof. Choose any $p, q \geqslant 2$, with $p + q = d$, and let $P = T^p \times T^q$ be the cartesian product of a p-simplex and a q-simplex. If we write vert $T^p = \{x_0, \ldots, x_p\}$ and vert $T^q = \{y_0, \ldots, y_q\}$, then the vertices of P are all (x_i, y_j), with $i = 0, \ldots, p$ and $j = 0, \ldots, q$. The two vertices (x_i, y_j) and $(x_{i'}, y_{j'})$ determine an edge of P if and only if $i = i'$ or $j = j'$, but not both.

Let us choose $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ as principal vertices. We shall show that there are exactly two refinements of C_{d+1} in $G(P)$ with z_0 and z_1 principal, and that neither contains $z_2 = (x_2, y_2)$ as a principal vertex. Hence, for no refinement of C_{d+1} are z_0, z_1 and z_2 all principal.

Let the desired refinement of C_{d+1} with z_0 and z_1 principal be the subgraph B of $G(P)$. Since both T^p and T^q are simple, so is P, and hence exactly d edges meet at each vertex. At each principal vertex of B, d edges also meet. Hence, B contains every edge of P through z_0 or z_1. Let B_0 be the subgraph of B consisting of these edges.

In B_0 two paths join z_0 and z_1, namely those through (x_0, y_1) and (x_1, y_0). At most one of these paths can correspond to an edge of C_{d+1}, and so at least one contains another principal vertex. So, let us suppose that (x_0, y_1) is principal. As before, B must contain all the edges of P through (x_0, y_1); let B_1 consist of these edges and those of B_0. Now, z_0 and (x_0, y_1) are joined by an edge of B_1, which must correspond to an edge of C_{d+1}. Thus every other path in B_1 joining z_0 and (x_0, y_1) must contain a principal vertex, and so the vertices (x_0, y_j) $(j = 2, \dots, q)$ are all principal.

In an exactly similar way, we see that all the vertices (x_i, y_1) $(i = 0, \dots, p)$ are principal. We have now found $p + q + 1 = d + 1$ principal vertices; that is, we have all of them. The graph B then consists of all the edges of P meeting these principal vertices; it is easily seen to be a refinement of C_{d+1}. For, any two principal vertices (x_0, y_j) and $(x_0, y_{j'})$ are joined by an edge of B, as are any two (x_i, y_1) and $(x_{i'}, y_1)$, while any pair (x_0, y_j) and (x_i, y_1), with $j \neq 1$ and $i \neq 0$, is joined by a path of two edges of B passing through the intermediate vertex (x_i, y_j), which is used by this pair alone.

If (x_1, y_0) is chosen as a principal vertex, we get the alternative refinement, with principal vertices (x_i, y_0) $(i = 0, \dots, p)$ and (x_1, y_j) $(j = 1, \dots, q)$. The two cases are distinct, but symmetrical. We also note that one case excludes the other; (x_0, y_1) and (x_1, y_0) cannot both be principal. Finally, we observe that in neither case is z_2 a principal vertex, and this establishes Theorem 2.

Theorem 2, together with Grünbaum's result, implies that for $d \geqslant 4$, $r(d) = 1$ or 2. Nothing further is known, although Larman and Mani suggest that $r(d) = 2$. It is tempting to generalize the fact that if $d \leqslant 3$, in a d-connected graph we may preassign two vertices as principal in a refiement of C_{d+1}. However, for $d \geqslant 4$, a d-connected graph need not contain a refinement of C_{d+1} at all. For instance, the graph of the regular icosahedron is 5-connected, yet, by Kuratowski's theorem, it contains no refinement of C_5, and hence none of C_6.

We observe, however, that the conjecture $r(d) = 2$ is, in some sense, a natural generalization of the fact that $G(P)$ is d-connected if P is a d-polytope. For, the d arcs in $G(P)$, which join two given vertices, form a subgraph which is a refinement of the graph H_{d+1} obtained from C_{d+1} by deleting the edges of a complete subgraph C_{d-1}. We can recover C_{d+1} from H_{d+1} by adjoining to it the edges spanned by its $d-1$ 2-valent vertices.

We now move on to simplicial d-polytopes; the following will apply for $d \geqslant 3$. We first have a result from [3].

Theorem 3. *Let P be a simplicial d-polytope, and let x_0 and x_1 be two vertices of P. Then $G(P)$ contains a refinement of C_{d+1}, with x_0 and x_1 as principal vertices.*

Proof. Let F be a facet of P containing x_1 but not x_0 (see [2, 3.1.8]), and let vert $F = \{x_1, \ldots, x_d\}$. Then there are d arcs in $G(P)$, disjoint except at x_0, joining x_0 to $x_1, \ldots, x_d$, since $G(P)$ is d-connected. These arcs, together with the edges of F, then form the required refinement.

Note that the proof, which is a little different from that of [3], actually yields a rather stronger result than was asked for.

We thus have $r_s(d) \geqslant 2$. Examples of simplicial d-polytopes P for which $r(P) \leqslant 3$ are easily found.

Theorem 4. *Let $d \geqslant 3$. Then there is a simplicial d-polytope P, with some four vertices which cannot all be principal for any refinement of C_{d+1} in $G(P)$.*

Proof. We construct P as follows. Let T^d be a d-simplex, with vert $T^d = \{x_0, \ldots, x_d\}$, and, for $i = 0, 1$, let F_i be the facet of T^d which does not contain x_i. Let P be obtained from T^d by erecting over each F_i a shallow pyramid with apex y_i.

Suppose that C is a refinement of C_{d+1} in $G(P)$, with y_0 and y_1 both principal. Since C_{d+1} is d-valent at each vertex, and y_0 and y_1 are d-valent vertices of $G(P)$, every edge of $G(P)$ which meets y_0 or y_1 belongs to C. Hence, for $j = 2, \ldots, d$, the arc which joins y_0 and y_1, and passes through x_j, lies in C. At most one of these arcs corresponds to an edge of C_{d+1} in the refinement, and so $d-2$ of these arcs each contain another principal vertex, which must come from $x_2, \ldots, x_d$. Together with y_0 and y_1, these account for d principal vertices. Since there are only $d+1$ principal vertices in all, we see that x_0 and x_1 cannot both be principal. Thus x_0, x_1, y_0 and y_1 cannot all be principal in any refinement of C_{d+1} in $G(P)$, which proves the theorem.

Combining Larman and Mani's result with Theorem 4 shows that for $d \geqslant 4$, $r_s(d) = 2$ or 3.

References

[1] B. Grünbaum, On the facial structure of convex polytopes, Bull. Amer. Math. Soc. 71 (1965) 559–560.

[2] B. Grünbaum. Convex Polytopes. (Wiley-Interscience, London–New York–Sydney, 1967).

[3] D.G. Larman and P. Mani, On the existence of certain configurations within graphs and the 1-skeletons of polytopes, Proc. London Math. Soc. 20 (3) (1970) 144–160.